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Analytic first integrals of the Halphen system

Clàudia Valls

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

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Abstract

In this paper we provide a complete description of the first integrals of the classical Halphen system that can be described by formal power series. As a corollary we also obtain a complete description of its analytic first integrals in a neighborhood of the origin.

MSC: 34C35; 34D30

Keywords: Halphen system; Analytic first integrals

1. Introduction to the problem

The Halphen system (see [2])

$$\dot{x}_1 = x_2x_3 - x_1(x_2 + x_3), \quad \dot{x}_2 = x_3x_1 - x_2(x_3 + x_1), \quad \dot{x}_3 = x_1x_2 - x_3(x_1 + x_2), \tag{1}$$

is a famous model (see for instance [2,4,5]), where x_1, x_2 and x_3 are real variables. This system first appeared in Darboux's work (see [2]) and was later solved by Halphen in [5]. Later it was shown that this system is equivalent to the Einstein field equations for a diagonal self-dual Bianchi-IX metric with Euclidean signature (see [1,4]) and also that arises in the similarity reductions of associativity equations on a three-dimensional Frobenius manifold

E-mail address: cvalls@math.ist.utl.pt.

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(see [3]). From the point of view of the integrability, this system has been intensively studied using different theories. One of the main results in this direction is that system (1) can be explicitly integrated, since we can express its general solution in terms of elliptic integrals (see [5,8,7]) but the first integrals are not global and are multi-valued non-algebraic functions (see [9]). Another result that we want to mention is [6] where the authors prove by means of the so-called Darboux polynomials that system (1) does not admit a non-constant algebraic first integral.

The aim of this paper is to study the existence of first integrals of system (1) that can be described by formal series.

A *formal first integral* $f = f(x)$ of system (1) is a formal power series in the variables x such that

$$\sum_{k=1}^{3} \frac{\partial f}{\partial x} F_i(x) = 0,$$

different from a polynomial. We recall that the polynomial first integrals of system (1) were studied in [6] where the authors proved that system (1) does not admit a non-constant polynomial first integral.

The first main result of this paper is:

Theorem 1. *System (1) does not have formal first integrals.*

Here an *analytic first integral* of system (1) is an analytic function which is constant over the trajectories of system (1) and it is different from a polynomial. The second main result of this paper is:

Theorem 2. *System (1) does not have analytic first integrals in a neighborhood of the origin.*

To descrive the method of proof, we first note that the three hyperplanes

$$H_1 := x_1 - x_2 = 0, \quad H_2 := x_1 - x_3 = 0 \quad \text{and} \quad H_3 := x_2 - x_3 = 0, \tag{2}$$

are invariant by the flow of system (1) and that if $f := f(x_1, x_2, x_3)$ is a formal first integral of system (1), then for each $i = 1, 2, 3$, the restriction of f to $H_i = 0$ is also a formal first integral of system (1) restricted to $H_i = 0$. Thus, the method of proof will consist in studying completely the formal integrability of the reduced system on each $H_i = 0$ to get exact information on the formal first integrals of the whole system (1).

The paper is organized as follows: In Section 2 we prove some auxiliary results needed to prove Theorems 1 and 2. In Section 3 we provide the proof of Theorems 1 and 2.

2. Auxiliary results

Lemma 3. *Let $f = f(x_1, x_2, x_3)$ be a formal power series such that in $x_l = x_j$, $l, j \in \{1, 2, 3\}$, $l \neq j$, we have $f(x_1, x_2, x_3)|_{x_l=x_j} = \bar{f}$, where $\bar{f}$ is a formal power series in the variables x_j, x_k with $k \in \{1, 2, 3\}$, $k \neq j$ and $k \neq l$. Then, there exists a formal series $g = g(x_1, x_2, x_3)$ such that $f = \bar{f} + (x_l - x_j)g$.*

Proof. We denote by $\mathbb{Z}^+$ the set of all non-negative integers. We write

$$f = \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}^+)^3} f_{k_1,k_2,k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3}.$$

Without loss of generality we can assume $l = 1$ and $j = 2$. Then, writing $x_1 = x_2 + (x_1 - x_2)$, we have

$$f = \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}^+)^3} f_{k_1,k_2,k_3} (x_2 + (x_1 - x_2))^{k_1} x_2^{k_2} x_3^{k_3}.$$

Now, by the Newton's binomial formula we have

$$\begin{aligned} f &= \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}^+)^3} f_{k_1,k_2,k_3} \sum_{j=0}^{k_1} \binom{k_1}{j} x_2^j (x_1 - x_2)^{k_1-j} x_2^{k_2} x_3^{k_3} \\ &= \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}^+)^3} f_{k_1,k_2,k_3} x_2^{k_1+k_2} x_3^{k_3} + (x_1 - x_2) \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}^+)^3} f_{k_1,k_2,k_3} \sum_{j=0}^{k_1-1} \binom{k_1}{j} \\ &\quad \times (x_1 - c_0)^{k_1-j-1} x_2^{j+k_2} x_3^{k_3} = f(x_2, x_3) + (x_1 - x_2) g(x_1, x_2, x_3) \\ &= \bar{f} + (x_1 - x_2) g, \end{aligned}$$

which finishes the proof of the lemma. □

Lemma 4. *Let f be a formal first integral of system (1). Then*

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g,$$

where c_0 is some constant and $g := g(x_1, x_2, x_3)$ is a formal power series.

Proof. Let f be a formal first integral of system (1). We will first prove that if we denote by $\bar{f}$ the restriction of f to $\{x_1 = x_2\}$, then $\bar{f} = c_0$. Indeed, since from (2), $\{x_1 = x_2\}$ is invariant by system (1), we have that $\bar{f} := \bar{f}(x_2, x_3) = f(x_2, x_2, x_3)$ is a formal first integral of system (1) restricted to $x_1 = x_2$, that is, of system

$$\dot{x}_2 = -x_2^2, \quad \dot{x}_3 = x_2^2 - 2x_2x_3.$$

Then, after simplifying by x_2, $\bar{f}$ satisfies

$$-x_2 \frac{\partial \bar{f}}{\partial x_2} + (x_2 - 2x_3) \frac{\partial \bar{f}}{\partial x_3} = 0. \tag{3}$$

We write $\overline{f}$ in formal power series in the variables x_2 and x_3 as

$$\overline{f} = \sum_{k,l\geq 0} \overline{f}_{k,l} x_2^k x_3^l. \tag{4}$$

Thus, imposing that $\overline{f}$ satisfies (3) we get that

$$-\sum_{k,l\geq 0}(k+2l)\overline{f}_{k,l}x_2^k x_3^l + \sum_{k,l\geq 0} l\overline{f}_{k,l}x_2^{k+1}x_3^{l-1} = 0,$$

which can be rewritten as

$$\sum_{k,l\geq 0}((k+2l)\overline{f}_{k,l} - (l+1)\overline{f}_{k-1,l+1})x_2^k x_3^l = 0, \tag{5}$$

where $\overline{f}_{m,n} = 0$ for $m < 0$. Therefore, (5) implies that

$$(k+2l)\overline{f}_{k,l} - (l+1)\overline{f}_{k-1,l+1} = 0 \quad \text{for every} k, l \geq 0. \tag{6}$$

Now, we claim that

$$\overline{f}_{k,l} = 0 \quad \text{for } k, l \geq 0 \quad \text{and} \quad (k, l) \neq (0, 0). \tag{7}$$

We will prove (7) by induction over k. For $k = 0$, (6) becomes $2l\overline{f}_{0,l} = 0$ for $l \geq 0$, which clearly implies $\overline{f}_{0,l} = 0$ for $l > 0$ and proves (7) for $k = 0$. Now, assume (7) is true for $k = 0, \ldots, m-1$ ($m \geq 1$) and we will prove it for $k = m$. By inductive hypothesis, (6) for $k = m$ is just $(m+2l)\overline{f}_{m,l} = 0$ for $l \geq 0$, and thus, $\overline{f}_{m,l} = 0$ for $l \geq 0$. Then, (7) is proved for $k = m$ and by induction, (7) holds. Then, from (4) and (7) we get that $\overline{f} = \overline{f}_{0,0} := c_0$ and, using Lemma 3 with $x_l = x_1$ and $x_j = x_2$, we obtain

$$f = c_0 + (x_1 - x_2)g_0, \tag{8}$$

for some formal power series $g_0 := g_0(x_1, x_2, x_3)$.

Now, we consider $\hat{f}$ the restriction of f to $\{x_1 = x_3\}$. Then, repeating for $\hat{f}$ the arguments we did for $\overline{f}$, we get that $\hat{f} = d_0$, where d_0 is some constant. Thus, applying again Lemma 3 with $x_l = x_1$ and $x_j = x_3$, we get that f can also be written as

$$f = d_0 + (x_1 - x_3)g_1, \tag{9}$$

for some formal power series $g_1 := g_1(x_1, x_2, x_3)$. In a similar way, taking $\tilde{f}$ the restriction of f to $\{x_2 = x_3\}$ and repeating for $\tilde{f}$ the arguments we did for $\overline{f}$ we get that $\tilde{f} = e_0$ for some constant e_0. Then, again using Lemma 3 with $x_l = x_2$ and $x_j = x_3$, we get that f can also be written as

$$f = e_0 + (x_2 - x_3)g_2, \tag{10}$$

for some formal power series $g_2 := g_2(x_1, x_2, x_3)$. Now, evaluating (8)–(10) to $x_1 = x_2 = x_3 = 0$ and equating them, we get that $e_0 = d_0 = c_0$. Furthermore, equating (8)–(10) we get

$$(x_1 - x_2)g_0 = (x_1 - x_3)g_1 = (x_2 - x_3)g_2,$$

which clearly implies that there exists a formal power series $g := g(x_1, x_2, x_3)$ such that

$$g_0 = (x_1 - x_3)(x_2 - x_3)g, \quad g_1 = (x_1 - x_2)(x_2 - x_3)g, \quad g_2 = (x_1 - x_2)(x_1 - x_3)g. \tag{11}$$

Therefore, the proposition follows from (8) and the first relation in (11). □

3. Proof of the main theorems

Proof of Theorem 1. Let f be any formal first integral of system (1). By Proposition 4 we know that f can be written as

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g, \tag{12}$$

for some constant c_0 and some formal power series $g := g(x_1, x_2, x_3)$. Imposing that f is a first integral of system (1), we get that, after simplifying by $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, g must satisfy

$$\frac{\mathrm{d}g}{\mathrm{d}t} = 2(x_3 + x_2 + x_1)g, \tag{13}$$

where the derivative is evaluated along a solution of system (1). We will prove that $g = 0$. For this, we will proceed by reduction to the absurd. Assume $g \neq 0$ and we will reach a contradiction. We consider two different cases.

Case 1: g is not divisible by $x_1 - x_2$. In this case, using Lemma 3 with $x_l = x_1$ and $x_j = x_2$, we can write g as $g = g_0 + (x_1 - x_2)g_1$, where $g_0 := g_0(x_2, x_3) \neq 0$ and $g_1 := g_1(x_1, x_2, x_3)$ are formal power series. Then, g_0 satisfies (13) restricted to $x_1 = x_2$, that is

$$-x_2\left(x_2\frac{\partial g_0}{\partial x_2} - (x_2 - 2x_3)\frac{\partial g_0}{\partial x_3}\right) = 2(x_3 + 2x_2)g_0. \tag{14}$$

Now, we write

$$g_0 = \sum_{j\geq 0} g_{0,j}x_2^j, \quad g_j = g_j(x_3) \quad \text{with}\, g_j \text{ being formal power series in}\, x_3.$$

We claim that

$$g_{0,j} = 0 \quad \text{for } j \geq 0. \tag{15}$$

From (14) we get that g_0 must be divisible by x_2 and thus, $g_{0,0} = 0$. Hence, Eq. (15) is proved for $j = 0$. Now, we assume (15) is true for $j = 0, \ldots, m-1$ ($m \geq 1$) and we will prove it for $j = m$. Clearly, by hypothesis of induction, $g_0 = \sum_{j\geq 0} g_{0,j+m}x_2^{j+m}$ and then, from (14), after dividing by x_2^m, we obtain

$$-2x_3\sum_{j\geq 0} g_{0,j+m}x_2^j - x_2\sum_{j\geq 0}(4 + j + m)g_{0,j+m}x_2^j + x_2(x_2 - 2x_3)\sum_{j\geq 0}\frac{\mathrm{d}g_{0,j+m}}{\mathrm{d}x_3}x_2^j = 0. \tag{16}$$

Then, evaluating (16) on $x_2 = 0$, we get $-2x_3 g_{0,m} = 0$, which clearly implies $g_{0,m} = 0$ and proves (15) for $j = m$. Then, by the induction process, (15) holds and from (15) we get that $g_0 = 0$, a contradiction. *Case 2*: *g is divisible by* $x_1 - x_2$. In this case, $g = (x_1 - x_2)^j h$ with $j \geq 1$, $h \neq 0$ and $h := h(x_1, x_2, x_3)$ is a formal power series such that is not divisible by $x_1 - x_2$ and satisfies, after dividing by $(x_1 - x_2)^j$, the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 2(x_1 + x_2 + (j+1)x_3)h,$$

where the derivative of h is evaluated along a solution of system (1). Then, applying to h the same arguments used for g in Case 1, we conclude that $h = 0$, a contradiction.

Hence, $g = 0$ and the proof of the theorem follows from (12) and the definition of formal first integral. □

Proof of Theorem 2. To prove that system (1) does not have analytic first integrals in a neighborhood of zero, we proceed by contradiction. Assume that g is an analytic first integral of system (1) in a neighborhood $U \subset \mathbb{R}^3$ of the origin. Clearly, $g_{|U}$ can be written as a formal power series which turns out to be convergent. Hence, in U, g is a formal first integral of system (1), a contradiction with Theorem 1. Thus, Theorem 2 is proved. □

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